

# A Class of Exponential Sequences with Shift-Invariant Discriminators

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## Abstract

The discriminator of an integer sequence  $\mathbf{s} = (s(i))_{i \geq 0}$ , introduced by Arnold, Benkoski, and McCabe in 1985, is the function  $D_{\mathbf{s}}(n)$  that sends  $n$  to the least integer  $m$  such that the numbers  $s(0), s(1), \dots, s(n-1)$  are pairwise incongruent modulo  $m$ . In this note we present a class of exponential sequences that have the special property that their discriminators are shift-invariant, i.e., that the discriminator of the sequence is the same even if the sequence is shifted by any positive constant.

## 1 Discriminators

Let  $m$  be a positive integer. If  $S$  is a set of integers that are pairwise incongruent modulo  $m$ , we say that  $m$  *discriminates*  $S$ . Now let  $\mathbf{s} = (s(i))_{i \geq 0}$  be a sequence of distinct integers. For all integers  $n \geq 1$ , we define  $D_{\mathbf{s}}(n)$  to be the least positive integer  $m$  that discriminates the set  $\{s(0), s(1), \dots, s(n-1)\}$ . The function  $D_{\mathbf{s}}(n)$  is called the *discriminator* of the sequence  $\mathbf{s}$ .

The discriminator was first introduced by Arnold, Benkoski, and McCabe [1]. They derived the discriminator for the sequence  $1, 4, 9, \dots$  of positive integer squares. More recently, discriminators of various sequences were studied by Schumer and Steinig [11], Barcau [2], Schumer [10], Bremser, Schumer, and Washington [3], Moree and Roskam [8], Moree [6], Moree and Mullen [7], Zieve [13], Sun [12], Moree and Zumalacárrequi [9], and Haque and Shallit [5].

In all of these cases, however, the discriminator is based on the first  $n$  terms of a sequence, for  $n \geq 2$ . Therefore, the discriminator can depend crucially on the starting point of a

given sequence. For example, although the discriminator for the first three positive squares,  $\{1, 4, 9\}$ , is 6, we can see that the number 6 does not discriminate the length-3 “window” into the shifted sequence,  $\{4, 9, 16\}$ , since  $16 \equiv 4 \pmod{6}$ .

Furthermore, there has been very little work on the discriminators of exponential sequences. Sun [12] presented some conjectures concerning certain exponential sequences, while in a recent *tour de force*, Moree and Zumalacárrequis [9] computed the discriminator for the sequence  $\left(\frac{|(-3)^j - 5|}{4}\right)_{j \geq 0}$ .

We say that the discriminator of a sequence is *shift-invariant* if the discriminator for the sequence is the same even if the sequence is shifted by any positive integer  $c$ , i.e., for all positive integers  $c$  the discriminator of the sequence  $(s(n))_{n \geq 1}$  is the same as the discriminator of the sequence  $(s(n+c))_{n \geq 0}$ . In this paper, we present a class of exponential sequences whose discriminators are shift-invariant.

We define this class of exponential sequences as follows:

$$(\text{ex}(n))_{n \geq 0} = \left(a \frac{(t^2)^n - 1}{2^b}\right)_{n \geq 0}$$

for odd positive integers  $a$  and  $t$ , where  $b$  is the smallest positive integer such that  $t \not\equiv \pm 1 \pmod{2^b}$ . A typical example is the sequence  $\left(\frac{9^n - 1}{8}\right)_{n \geq 0}$ . We show that the discriminator for all sequences of this form is  $D_{\text{ex}}(n) = 2^{\lceil \log_2 n \rceil}$ . Furthermore, we show that this discriminator is shift-invariant, i.e., it applies to every sequence  $(\text{ex}(n+c))_{n \geq 0}$  for  $c \geq 0$ .

The outline of the paper is as follows. In Section 2 we obtain an upper bound for the discriminator of  $\left(\frac{(t^2)^n - 1}{2^b}\right)_{n \geq 0}$  and all of its shifts. In Section 3 we prove some lemmas that are essential to our lower bound proof. Finally, in Section 4 we put the results together to determine the discriminator for  $\left(a \frac{(t^2)^n - 1}{2^b}\right)_{n \geq 0}$  and all of its shifts.

## 2 Upper bound

In this section, we derive an upper bound for the discriminator of the sequence  $\left(\frac{(t^2)^n - 1}{2^b}\right)_{n \geq 0}$  and all of its shifts. We start with some useful lemmas.

**Lemma 1.** *Let  $t$  be an odd integer, and let  $b$  be the smallest positive integer such that  $t \not\equiv \pm 1 \pmod{2^b}$ . Then  $t^2 \equiv 2^b + 1 \pmod{2^{b+1}}$ .*

*Proof.* Note that since every odd integer equals  $\pm 1$  modulo 4, we must have  $b \geq 3$ . From the definition of  $b$ , we have  $t \equiv 2^{b-1} \pm 1 \pmod{2^b}$ . Hence  $t = 2^b c + 2^{b-1} \pm 1$  for some integer  $c$ . By squaring both sides of the equation, we get

$$\begin{aligned} t^2 &= 2^{2b} c^2 + 2^{2(b-1)} + 2^{2b} c \pm 2^{b+1} c \pm 2^b + 1 \\ &= 2^{b+1} (2^{b-1} c^2 + 2^{b-3} + 2^{b-1} c \pm c) \pm 2^b + 1, \\ \implies t^2 &\equiv \pm 2^b + 1 \pmod{2^{b+1}}, \\ \implies t^2 &\equiv 2^b + 1 \pmod{2^{b+1}}. \end{aligned}$$

□

**Lemma 2.** *Let  $t$  be an odd integer, and let  $b$  be the smallest positive integer such that  $t \not\equiv \pm 1 \pmod{2^b}$ . Then we have*

$$t^{2^k} \equiv 2^{k+b-1} + 1 \pmod{2^{k+b}} \quad (1)$$

for all integers  $k \geq 1$ .

*Proof.* By induction on  $k$ .

**Base case:** From Lemma 1, we have  $t^2 \equiv 2^b + 1 \pmod{2^{b+1}}$ .

**Induction:** Suppose Eq. (1) holds for some  $k \geq 1$ , i.e.,  $t^{2^k} \equiv 2^{k+b-1} + 1 \pmod{2^{k+b}}$ . This means that  $t^{2^k} = 2^{k+b}c + 2^{k+b-1} + 1$  for some integer  $c$ . Once again, by squaring both sides of the equation, we get

$$\begin{aligned} \left(t^{2^k}\right)^2 &= t^{2^{k+1}} = 2^{2k+2b}c^2 + 2^{2k+2b-2} + 1 + 2^{2k+2b}c + 2^{k+b+1}c + 2^{k+b} \\ &= 2^{k+b+1} \left(2^{k+b-1}c^2 + 2^{k+b-3} + 2^{k+b-1}c + c\right) + 2^{k+b} + 1, \\ \implies t^{2^{k+1}} &\equiv 2^{k+b} + 1 \pmod{2^{k+b+1}}. \end{aligned}$$

This shows that Eq. (1) holds for  $k+1$  as well, thus completing the induction.

□

This gives the following corollary.

**Corollary 3.** *Let  $t$  be an odd integer, and let  $b$  be the smallest positive integer such that  $t \not\equiv \pm 1 \pmod{2^b}$ . Then for  $k \geq 1$ , the powers of  $t^2$  form a cyclic subgroup of order  $2^k$  in  $(\mathbb{Z}/2^{k+b})^*$ .*

*Proof.* Let  $\ell = k+1$ . Since  $\ell \geq 1$ , we can apply Eq. (1) to get

$$\begin{aligned} (t^2)^{2^{\ell-1}} &= t^{2^\ell} \equiv 2^{\ell+b-1} + 1 \pmod{2^{\ell+b}}, \\ \implies (t^2)^{2^{\ell-1}} &\equiv 1 \pmod{2^{\ell+b-1}}, \\ \implies (t^2)^{2^k} &\equiv 1 \pmod{2^{k+b}}. \end{aligned}$$

Furthermore, by applying Eq. (1) directly, we get

$$\begin{aligned} (t^2)^{2^{k-1}} &= t^{2^k} \equiv 2^{k+b-1} + 1 \not\equiv 1 \pmod{2^{k+b}}, \\ \implies (t^2)^{2^{k-1}} &\not\equiv 1 \pmod{2^{k+b}}. \end{aligned}$$

Therefore, the order of the subgroup generated by  $t^2$  in  $(\mathbb{Z}/2^{k+b})^*$  is  $2^k$ .

□

**Lemma 4.** *Let  $t$  be an odd integer, and let  $b$  be the smallest positive integer such that  $t \not\equiv \pm 1 \pmod{2^b}$ . Then for  $k \geq 0$ , the number  $2^k$  discriminates every set of  $2^k$  consecutive terms of the sequence  $\left(\frac{(t^2)^n - 1}{2^b}\right)_{n \geq 0}$ .*

*Proof.* For every  $i \geq 0$ , it follows from Corollary 3 that the numbers

$$(t^2)^i, (t^2)^{i+1}, \dots, (t^2)^{i+2^k-1}$$

are distinct modulo  $2^{k+b}$ . By subtracting 1 from every element, we have that the numbers

$$(t^2)^i - 1, (t^2)^{i+1} - 1, \dots, (t^2)^{i+2^k-1} - 1$$

are distinct modulo  $2^{k+b}$ . Furthermore, these numbers are also congruent to 0 modulo  $2^b$  because  $t^2 \equiv 1 \pmod{2^b}$  from Lemma 1. It follows that the set of quotients

$$\left\{ \frac{(t^2)^i - 1}{2^b}, \frac{(t^2)^{i+1} - 1}{2^b}, \dots, \frac{(t^2)^{i+2^k-1} - 1}{2^b} \right\}$$

consists of integers that are distinct modulo  $\frac{2^{k+b}}{2^b} = 2^k$ .

Such a set of quotients coincides with every set of  $2^k$  consecutive terms of the sequence  $\left(\frac{(t^2)^n - 1}{2^b}\right)_{n \geq 0}$ . Since the numbers in each set are distinct modulo  $2^k$ , the desired result follows.  $\square$

### 3 Lower bound

In this section, we establish some results useful for the lower bound on the discriminator of the sequence  $\left(\frac{(t^2)^n - 1}{2^b}\right)_{n \geq 0}$ . We start with an easy technical lemma, whose proof is omitted.

**Lemma 5.** *Let  $m$  be a positive integer. Then  $\log_3 m \leq \frac{m}{3}$ .*

The main lemma for proving the lower bound is as follows:

**Lemma 6.** *Let  $t$  be an odd integer, and let  $b$  be the smallest positive integer such that  $t \not\equiv \pm 1 \pmod{2^b}$ . Then for all  $k \geq 0$  and  $1 \leq m \leq 2^{k+1}$ , there exists a pair of integers,  $i$  and  $j$ , where  $0 \leq i < j \leq 2^k$ , such that  $(t^2)^i \equiv (t^2)^j \pmod{2^b m}$ .*

*Proof.* Let the prime factorization of  $m$  be

$$m = 2^x \prod_{1 \leq \ell \leq u} p_\ell^{y_\ell} \prod_{1 \leq \ell \leq v} q_\ell^{z_\ell},$$

where  $u, v, x, y_\ell, z_\ell \geq 0$ , while  $p_1, p_2, \dots, p_u$  are the prime factors of  $m$  that also divide  $t$ , and  $q_1, q_2, \dots, q_v$  are the odd prime factors of  $m$  that do not divide  $t$ . For each  $\ell \leq u$ , let  $e_\ell$  be the integer such that  $p_\ell^{e_\ell} \parallel t$ , i.e., we have  $p_\ell^{e_\ell} \mid t$  but  $p_\ell^{e_\ell+1} \nmid t$ .

We need to find a pair  $(i, j)$  such that  $(t^2)^i \equiv (t^2)^j \pmod{2^b m}$ . From the Chinese remainder theorem, we know it suffices to find a pair  $(i, j)$  such that

$$\begin{aligned} (t^2)^i &\equiv (t^2)^j \pmod{2^{x+b}}, \\ (t^2)^i &\equiv (t^2)^j \pmod{p_\ell^{y_\ell}}, \text{ for all } 1 \leq \ell \leq u, \\ \text{and } (t^2)^i &\equiv (t^2)^j \pmod{q_\ell^{z_\ell}}, \text{ for all } 1 \leq \ell \leq v. \end{aligned}$$

For the first of these equations, we know from Corollary 3 that  $(t^2)^i \equiv (t^2)^{i+2^x} \pmod{2^{x+b}}$ . In other words, it suffices to have  $2^x | (j - i)$  to satisfy  $(t^2)^i \equiv (t^2)^j \pmod{2^{x+b}}$ .

Next, we consider the  $u$  equations of the form  $(t^2)^i \equiv (t^2)^j \pmod{p_\ell^{y_\ell}}$ . Since  $p_\ell^{e_\ell}$  is a factor of  $t$ , it follows that  $(t^2)^{y_\ell/2e_\ell}$  is a multiple of  $(p_\ell^{2e_\ell})^{y_\ell/2e_\ell} = p_\ell^{y_\ell}$ . Therefore,  $(t^2)^{y_\ell/2e_\ell} \equiv 0 \pmod{p_\ell^{y_\ell}}$ . Any further multiplication by  $t^2$  also yields 0 modulo  $p_\ell^{y_\ell}$ . Thus, it suffices to have  $j > i \geq \frac{y_\ell}{2e_\ell}$  in order to ensure that  $(t^2)^i \equiv (t^2)^j \pmod{p_\ell^{y_\ell}}$ .

Finally, there are  $v$  equations of the form  $(t^2)^i \equiv (t^2)^j \pmod{q_\ell^{z_\ell}}$ . In each case,  $q_\ell$  is co-prime to  $t$ , which means that  $(t^2)^{\varphi(q_\ell^{z_\ell})/2} = t^{\varphi(q_\ell^{z_\ell})} \equiv 1 \pmod{q_\ell^{z_\ell}}$ , where  $\varphi(n)$  is Euler's totient function. Now  $\frac{\varphi(q_\ell^{z_\ell})}{2} = \frac{q_\ell^{z_\ell-1}(q_\ell-1)}{2}$ . Thus, it is sufficient to have  $\frac{q_\ell^{z_\ell-1}(q_\ell-1)}{2} | (j - i)$  in order to ensure that  $(t^2)^i \equiv (t^2)^j \pmod{q_\ell^{z_\ell}}$ .

Merging these ideas together, we choose the following values for  $i$  and  $j$ :

$$\begin{aligned} i &= \max_{1 \leq \ell \leq u} \left\lceil \frac{y_\ell}{2e_\ell} \right\rceil, \\ j &= \max_{1 \leq \ell \leq u} \left\lceil \frac{y_\ell}{2e_\ell} \right\rceil + 2^x \prod_{1 \leq \ell \leq v} \frac{q_\ell^{z_\ell-1}(q_\ell-1)}{2}, \end{aligned}$$

to ensure that  $(t^2)^i \equiv (t^2)^j \pmod{2^b m}$ . It is clear that  $0 \leq i < j$ . In order to show that  $j \leq 2^k$ , we first observe that

$$\begin{aligned} j &= \max_{1 \leq \ell \leq u} \left\lceil \frac{y_\ell}{2e_\ell} \right\rceil + 2^x \prod_{1 \leq \ell \leq v} \frac{q_\ell^{z_\ell-1}(q_\ell-1)}{2} = \max_{1 \leq \ell \leq u} \left\lceil \frac{y_\ell}{2e_\ell} \right\rceil + \frac{2^x}{2^v} \prod_{1 \leq \ell \leq v} q_\ell^{z_\ell-1}(q_\ell-1) \\ &\leq \max_{1 \leq \ell \leq u} \left\lceil \frac{y_\ell}{2} \right\rceil + \frac{2^x}{2^v} \prod_{1 \leq \ell \leq v} q_\ell^{z_\ell} = \max_{1 \leq \ell \leq u} \left\lceil \frac{y_\ell}{2} \right\rceil + \frac{m}{2^v \prod_{1 \leq \ell \leq u} p_\ell^{y_\ell}}. \end{aligned}$$

We now consider the following two cases:

**Case 1:**  $u = 0$ . If  $v = 0$  as well, then  $j = 2^x = m < 2^{k+1}$ , which means that  $x \leq k$  and thus  $j \leq 2^k$ . Otherwise, if  $v \geq 1$ , then we have

$$j \leq \max_{1 \leq \ell \leq u} \left\lceil \frac{y_\ell}{2} \right\rceil + \frac{m}{2^v \prod_{1 \leq \ell \leq u} p_\ell^{y_\ell}} = \frac{m}{2^v} \leq \frac{m}{2} < \frac{2^{k+1}}{2} = 2^k.$$

**Case 2:**  $u \geq 1$ . Let  $r$  be such that  $y_r = \max_{1 \leq \ell \leq u} y_\ell$ , and thus,  $p_r$  is the corresponding prime number with exponent  $y_r$ . Since  $p_r^{y_r} \geq p_r \geq 3$ , we have

$$j \leq \max_{1 \leq \ell \leq u} \left\lceil \frac{y_\ell}{2} \right\rceil + \frac{m}{2^v \prod_{1 \leq \ell \leq u} p_\ell^{y_\ell}} \leq \left\lceil \frac{y_r}{2} \right\rceil + \frac{m}{p_r^{y_r}} \leq \frac{y_r + 1}{2} + \frac{m}{3} \leq \frac{y_r}{2} + \frac{1}{2} + \frac{m}{3}.$$

Note that  $y_r \leq \log_{p_r} m \leq \log_3 m \leq \frac{m}{3}$  from Lemma 5, which means that

$$j \leq \frac{y_r}{2} + \frac{1}{2} + \frac{m}{3} \leq \frac{m}{6} + \frac{1}{2} + \frac{m}{3} = \frac{m}{2} + \frac{1}{2} = \frac{m+1}{2}.$$

Since both  $m$  and  $j$  are integers, this implies that

$$j \leq \left\lceil \frac{m}{2} \right\rceil \leq \left\lceil \frac{2^{k+1}}{2} \right\rceil \leq 2^k.$$

In both cases, we have  $j \leq 2^k$ , thus fulfilling the required conditions.  $\square$

## 4 Discriminator of $(\text{ex}(n))_{n \geq 0}$ and its shifted counterparts

In this section, we combine the results of the previous sections to determine the discriminator for  $(\text{ex}(n))_{n \geq 1}$ , as well as its shifted counterparts. We first prove a general lemma about the discriminator of some scaled sequences.

**Lemma 7.** *Given a sequence  $s(0), s(1), \dots$ , and a non-zero integer  $a$ , let  $s'(0), s'(1), \dots$ , denote the sequence such that  $s'(i) = as(i)$  for all  $i \geq 0$ . Then, for every  $n$  such that  $\gcd(|a|, D_s(n)) = 1$ , we have  $D_{s'}(n) = D_s(n)$ .*

*Proof.* From the definition of the discriminator, we know that for every  $m < D_s(n)$ , there exists a pair of integers  $i$  and  $j$  with  $i < j < n$ , such that  $m \nmid s(j) - s(i)$ . Thus, for this same pair of  $i$  and  $j$ , we have

$$m \nmid a(s(j) - s(i)) = as(j) - as(i) = s'(j) - s'(i).$$

Therefore,  $m$  cannot discriminate the set  $\{s'(0), s'(1), \dots, s'(n-1)\}$  and so  $D_{s'}(n) \geq D_s(n)$ .

But for  $m = D_s(n)$ , we know that for all  $i$  and  $j$  with  $i < j < n$ , we have  $m \nmid s(j) - s(i)$ . Since  $\gcd(m, |a|) = 1$ , it follows that

$$m \nmid a(s(j) - s(i)) = as(j) - as(i) = s'(j) - s'(i)$$

for all  $i$  and  $j$  with  $i < j < n$ . Therefore,  $m = D_s(n)$  discriminates the set

$$\{s'(0), s'(1), \dots, s'(n-1)\}$$

and so  $D_{s'}(n) \leq D_s(n)$ .

Putting these results together, we have  $D_{s'}(n) = D_s(n)$ .  $\square$

We now compute the discriminator for  $(\text{ex}(n))_{n \geq 0} = \left(a \frac{(t^2)^n - 1}{2^b}\right)_{n \geq 0}$ , and also for its shifted counterparts, which we denote by  $(\text{exs}(n, c))_{n \geq 0} = (\text{ex}(n + c))_{n \geq 0}$  for some integer  $c \geq 0$ .

**Theorem 8.** *Let  $t$ ,  $a$ ,  $b$ , and  $c$  be integers such that  $a$  and  $t$  are odd,  $c \geq 0$ , and let  $b$  be the smallest integer such that  $t \not\equiv \pm 1 \pmod{2^b}$ . Then the discriminator for the sequence  $(\text{exs}(n, c))_{n \geq 0} = \left(a \frac{(t^2)^{n+c} - 1}{2^b}\right)_{n \geq 0}$  is*

$$D_{\text{exs}}(n) = 2^{\lceil \log_2 n \rceil}. \quad (2)$$

*Proof.* First we compute the discriminator for  $a = 1$ , where the sequence is of the form  $(\text{exs}(n))_{n \geq 0} = \left(\frac{(t^2)^{n+c} - 1}{2^b}\right)_{n \geq 0}$ .

The case for  $n = 1$  is trivial. Otherwise, let  $k \geq 0$  be such that  $2^k < n \leq 2^{k+1}$ . We show that  $D_{\text{exs}}(n) = 2^{k+1}$ .

From Lemma 4, we know that  $2^{k+1}$  discriminates the set,

$$\{\text{ex}(c), \text{ex}(c+1), \dots, \text{ex}(c+2^{k+1}-1)\},$$

as well as every smaller subset of these numbers. Therefore,  $2^{k+1}$  discriminates

$$\{\text{exs}(0, c), \text{exs}(1, c), \dots, \text{exs}(n-1, c)\}.$$

In other words,  $D_{\text{exs}}(n) \leq 2^{k+1}$ .

Now let  $m$  be a positive integer such that  $m < 2^{k+1}$ . By Lemma 6, we know that there exists a pair of integers,  $i$  and  $j$ , such that

$$\begin{aligned} (t^2)^i &\equiv (t^2)^j \pmod{2^b m} \implies (t^2)^c (t^2)^i \equiv (t^2)^c (t^2)^j \pmod{2^b m}, \\ &\implies (t^2)^{i+c} - 1 \equiv (t^2)^{j+c} - 1 \pmod{2^b m}. \end{aligned}$$

Note that since  $(t^2) \equiv 1 \pmod{2^b}$  from Lemma 1, we have  $(t^2)^{i+c} - 1 \equiv (t^2)^{j+c} - 1 \equiv 1 - 1 \equiv 0 \pmod{2^b}$ . Therefore,

$$(t^2)^{i+c} - 1 \equiv (t^2)^{j+c} - 1 \pmod{2^b m} \implies \frac{(t^2)^{i+c} - 1}{2^b} \equiv \frac{(t^2)^{j+c} - 1}{2^b} \pmod{m}.$$

In other words,  $\text{exs}(i, c) \equiv \text{exs}(j, c) \pmod{m}$  while both numbers are in the set

$$\{\text{exs}(0, c), \text{exs}(1, c), \dots, \text{exs}(n-1, c)\}$$

since  $i < j \leq 2^k < n$ . Therefore,  $m$  fails to discriminate this set. Since this applies for all  $m < 2^{k+1}$ , we have  $D_{\text{exs}}(n) \geq 2^{k+1}$ .

Since we have  $2^{k+1} \leq D_{\text{exs}} \leq 2^{k+1}$ , this means that  $D_{\text{exs}}(n) = 2^{k+1}$  and thus  $D_{\text{exs}}(n) = 2^{\lceil \log_2 n \rceil}$ , provided that  $a = 1$ .

Even for  $a \neq 1$ , we observe that the value of  $2^{\lceil \log_2 n \rceil}$  is a power of 2 for all  $n$ , and so it is co-prime to all odd  $a$ . Therefore, we can apply Lemma 7 to prove that the discriminator remains unchanged for odd values of  $a$ , thus proving that the discriminator for the sequence,  $(\text{exs}(n, c))_{n \geq 0} = \left(a \frac{(t^2)^{n+c} - 1}{2^b}\right)_{n \geq 0}$  is  $D_{\text{exs}}(n) = 2^{\lceil \log_2 n \rceil}$ .  $\square$

## 5 Final remarks

We have considered sequences of the form  $(\text{ex}(n))_{n \geq 0} = \left(a \frac{(t^2)^n - 1}{2^b}\right)_{n \geq 0}$  for odd integers  $a$  and  $t$ , where  $b$  is the smallest positive integer such that  $t \not\equiv \pm 1 \pmod{2^b}$ . We showed that the discriminator for this sequence is characterized by  $D_{\text{ex}}(n) = 2^{\lceil \log_2 n \rceil}$  and that the discriminator is shift-invariant, i.e., all sequences of the form  $(\text{ex}(n + c))_{n \geq 0}$  for  $c \geq 0$  share the same discriminator.

This raises the question of what other sequences have shift-invariant discriminators. It is easy to show that sequences defined by a linear equation, i.e. of the form  $(an + b)_{n \geq 0}$ , have shift-invariant discriminators. Furthermore, the first author has recently shown [4] that the sequence  $(2^k cn^2 + bcn)_{n \geq 0}$ , for a positive integer  $k$  and odd integers  $b, c$ , also has a shift-invariant discriminator  $2^{\lceil \log_2 n \rceil}$ .

It is an open problem as to whether there are any sequences, other than those mentioned here, whose discriminators are shift-invariant. Furthermore, all sequences whose discriminators are known to be shift-invariant have discriminators with linear growth, but we do not know if this is true of all shift-invariant discriminators.

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